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RATIONAL APPROXIMATIONS TO THE RESPONSE OF A DYNAMIC SYSTEM DESCRIBED BY A NONLINEAR DIFFERENTIAL EQUATION

by

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PREFACE

This report covers research initiated by Headquarters, National Aeronautics and Space Administration on Contract NASA Hq. R&D 80X0108(64), 10-74-740-124-08-06-11, PR 10-2487, "Nonlinear Dynamics of Thin Shell Structures." The research work upon which this report is based was accomplished at Midwest Research Institute with Mr. Howard Wolko as project monitor.

This report covers work conducted from January 15, 1964, to January 14, 1965.

The authors take this opportunity to thank Mrs. Geraldine Coombs and Miss Rosemary Moran for their assistance with the calculations. Special thanks are also due Mr. Jet Wimp for some valuable suggestions.

Approved for:

MIDWEST RESEARCH INSTITUTE

S L Levy

Sheldon L. Levy, Director
Mathematics and Physics Division

10 February 1965

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SUMMARY AND INTRODUCTION

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In this report we use a linear fractional transformation to obtain rational approximations to the response of a physical system which is defined by a second order nonlinear differential equation with constant coefficients. The initial approximations are constructed in the absence of damping. These approximations are then extended to include the case of a small damping term. The approximations are particularly effective in the case of slight damping and a large nonlinear restoring force.

In Section I we develop the recurrence relations which define the approximations. Section II has the treatment of a small damping term and also gives examples of the validity of the approximations.

Author 3

I. THE MASS SPRING OSCILLATOR EQUATION IN THE ABSENCE OF DAMPING

The mass spring oscillator equation

$$y'' + ay + by^2 + cy^3 + d = 0 \ , \ y(0) = a_0 \ , \ y'(0) = b_0 \tag{1.1}$$

where a, b, c and d are constants is well known. The classical approximations to (1.1) are of limited use. They are not easily constructed and little is known of their convergence properties. In the solution of numerous linear problems, approximation by rational functions, that is, the ratio of two polynomials, has proved very effective. A natural question is to see if a like procedure can be used for classes of nonlinear equations. As a first step in this direction, it is of interest to examine rational approximations to (1.1).

If the solution of a differential equation has a singularity, the radius of convergence of the Taylor's series solution about $x = a$ cannot exceed the distance from a to the singularity. Obviously, integration schemes based on polynomials are inadequate near a singularity. Approximations based on rational functions, on the other hand, show a greater degree of flexibility. If the singularity is a pole, as in (1.1), the rational function will mimic the behavior of the solution near the pole so effectively that often the position of the pole can be quite accurately determined from the approximation. We have also found such approximations to be effective in the neighborhood of other singularities, such as movable branch points.

Since the phenomenon of movable critical points is one of the biggest impediments to the approximate solution of nonlinear differential equations, the technique studied in this report, as well as the idea of rational approximations in general, may be fruitfully extended to large classes of nonlinear problems.

II. A GENERALIZED MASS SPRING OSCILLATOR EQUATION

It is convenient to generalize the problem by studying the first order equation

$$P_0(y')^2 + Q_0yy' + R_0y^2y' + S_0y' + T_0y + V_0y^2 + W_0y^3 + Y_0y^4 + X_0 = 0 \ ,$$
$$y(0) = a_0 \ , \tag{2.1}$$

where the coefficients in (2.1) are polynomials in x. Observe that (2.1) can be specialized to give (1.1), since the coefficients in (1.1) are constants.

In Ref. 1, Merkes and Scott developed continued fraction expansions for the solution to the Ricatti equation by using a sequence of linear fractional transformations. In Ref. 2, Fair constructed the main diagonal Padé approximations to the solution of the Ricatti equation by employing the τ-method. For details on the τ-method see Refs. 3 and 4.

In this section we utilize the linear fractional transformation to develop rational approximations to the response of a dynamical system described by the nonlinear differential equation (2.1). We assume that (2.1) has a series solution of the form

$$y = \sum_{k=0}^{\infty} c_k x^k \quad , \tag{2.2}$$

and further that y possesses a continued fraction representation of the form

$$y = \cfrac{a_0}{1 + \cfrac{a_1 x}{1 + \cfrac{a_2 x}{1 + \ddots}}} \quad , \tag{2.3}$$

which, by Ref. 5, is true if

$$d_{2m} = \begin{vmatrix} c_0 & 0 & \cdots\cdots & 0 & c_{2m-1} \\ 0 & c_1 & & c_{2m-1} & 0 \\ 0 & 0 & & 0 & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & 0 & & \cdot & \cdot \\ 0 & c_{2m-1} & 0 \cdot \cdot & c_{2m-1} & 0 \\ c_{2m-1} & 0 & \cdots\cdots & 0 & c_{2m} \end{vmatrix} \neq 0$$

(Eq. (2.4) concluded next page)

and

$$d_{2m+1} = \begin{vmatrix} c_o & 0 & \cdots & & 0 \\ 0 & c_1 & 0 & & \cdot \\ \cdot & 0 & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & 0 \\ 0 & \cdots & & & c_{2m+1} \end{vmatrix} \neq 0$$

$$m = 0,1,2, \ldots \tag{2.4}$$

For the general development of the theory of continued fractions and the relations defining their approximants, see Ref. 5.

If the continued fraction (2.3) is truncated, there results a rational approximation to the solution of (2.1). Generally, the sequence of convergents of the continued fraction converges much faster than the sequence of partial sums of the power series representation (2.2). This is especially true when the function has a pole near the origin.

Now the even approximants of (2.3) are known as the main diagonal Padé approximations which have the following properties. Let

$$y_n = \frac{A_n}{B_n} = \frac{\sum_{k=0}^{n} a_{n,k}x^k}{\sum_{k=0}^{n} b_{n,k}x^k} \tag{2.5}$$

be the n^{th} order main diagonal Padé approximant. Then if B_n is formally divided into A_n the resulting power series agrees with the power series solution for the first (2n+1) terms. The polynomials A_n and B_n both satisfy the relation

$$A_n = \left[1+(a_{2n-1}+a_{2n})x\right]A_{n-1} - a_{2n-1}a_{2n-2}x^2A_{2n-2} \quad ,$$

$$A_o = a_o \; , \; A_1 = a_o(1+a_1x) \; , \; B_o = 1 \;\text{ and }\; B_1 = 1 + (a_1+a_2)x \quad . \tag{2.6}$$

Once the values a_n , $n = 0,1,2,\ldots$ are determined, the Padé approximations (2.5) are readily computed. We present an algorithm to determine the a_n's .

Define the transformation

$$y_n = a_n(1+xy_{n+1})^{-1} \ , \ y_o = y \ , \ a_n = y_n(0) \quad . \tag{2.7}$$

We assume that in (2.1)

$$P_o(0) = Q_o(0) = R_o(0) = S_o(0) = V_o(0) = W_o(0) = Y_o(0) \quad ,$$

$$T_o(0) \neq 0 \quad \text{and} \quad X_o(0) \neq 0 \quad . \tag{2.8}$$

If (2.8) is not satisfied, one or two preliminary transformations of the type (2.7) result in an equation of the same form as (2.1) in which (2.8) is satisfied.

Successive application of (2.7) to (2.1) and division by x at each step gives

$$P_n(y_n')^2 + Q_n y_n y_n' + R_n y_n^2 y_n' + S_n y_n' + T_n y_n + V_n y_n^2 + W_n y_n^3 + Y_n y_n^4 + X_n = 0 \ , \tag{2.9}$$

where the polynomial coefficients in (2.9) are given by

$$P_{n+1} = a_n^2 x P_n \quad ,$$

$$Q_{n+1} = a_n\left[2a_nP_n - a_nxQ_n - 2xS_n\right],$$

$$R_{n+1} = - a_n x^2 S_n \quad ,$$

$$S_{n+1} = - a_n\left[a_nQ_n + a_n^2R_n + S_n\right] ,$$

(Eq. (2.10) concluded next page)

$$T_{n+1} = - x^{-1}\left[a_n^2Q_n+a_n^3R_n+a_nS_n-3a_nxT_n-2a_n^2xV_n-a_n^3xW_n-4xX_n\right] ,$$

$$V_{n+1} = x^{-1}\left[a_n^2P_n-a_n^2xQ_n-2a_nxS_n+3a_nx^2T_n+a_n^2x^2V_n+6x^2X_n\right] ,$$

$$W_{n+1} = x \left[-a_nS_n+a_nxT_n+4xX_n\right] ,$$

$$Y_{n+1} = x^3X_n ,$$

and

$$X_{n+1} = x^{-1}\left[a_nT_n+a_n^2V_n+a_n^3W_n+X_n+a_n^4Y_n\right] . \tag{2.10}$$

One can show by induction that all the polynomials in (2.10) are defined at $x = 0$, and that all the polynomials are zero at $x = 0$ except X_{n+1} and T_{n+1} . Further a_{n+1} is given by

$$a_{n+1} = y_{n+1}(0) = - \frac{X_{n+1}(0)}{T_{n+1}(0)} , \ n = 1,2,3,\ldots \tag{2.11}$$

These values of a_n determine the continued fraction (2.3) and hence the main diagonal Padé approximations (2.5).

The following example shows the power of these approximations. Consider the mass spring system with cubical stiffness,

$$y'' + 10y + 100y^3 = 0 , \ y(0) = 1 , \ y'(0) = 0 . \tag{2.12}$$

The solution to (2.12) is given by the Jacobian elliptic function $y = cn(n,k)$ where $u = \sqrt{110}\, t$ and $k^2 = 5/11$.

Tables I and II compare the values of the Padé approximants and the corresponding partial sums of the power series solution to the true solution. Note that, since the solution to (2.12) has its smallest pole at approximately $u_o = 1.9i$, the power series is ineffective for computation if its argument has magnitude close to 1.9, while the Padé approximations are very efficient as is illustrated by the tables.

TABLE I

MAIN DIAGONAL PADÉ

u	y (True)	$y_1(t)$	$y_2(t)$	$y_3(t)$
0.1751	0.9848	0.9848	0.9848	0.9848
0.3524	0.9396	0.9397	0.9396	0.9396
0.5349	0.8658	0.8660	0.8658	0.8658
0.7247	0.7655	0.7662	0.7655	0.7655
0.9240	0.6417	0.6444	0.6417	0.6417
1.1327	0.4993	0.5070	0.4993	0.4994
1.3550	0.3399	0.3585	0.3397	0.3399
1.5837	0.1724	0.2108	0.1713	0.1724
1.8238	-0.0040	0.0668	-0.0077	-0.0041

TABLE II

TAYLOR'S SERIES EXPANSION

u	y (True)	Three Terms	Five Terms	Seven Terms
0.1751	0.9848	0.9848	0.9848	0.9848
0.3524	0.9396	0.9397	0.9396	0.9396
0.5349	0.8658	0.8666	0.8658	0.8658
0.7247	0.7655	0.7698	0.7656	0.7655
0.9240	0.6417	0.6587	0.6427	0.6418
1.1327	0.4993	0.5518	0.5061	0.5002
1.3550	0.3399	0.4778	0.3762	0.3495
1.5837	0.1724	0.4846	0.3258	0.2474
1.8238	-0.0040	0.6360	0.5501	0.4720

Although the Padé approximations are more powerful than the partial sums of the power series, they exhibit the same deficiency in that the accuracy of the approximations decreases as the argument increases. To obviate the need of constructing extremely high order approximations to insure the desired accuracy, we present a continuation technique which can be used to construct a sequence of low order approximations that continues the solution from one interval to another.

Let $y_{n,0}$ be an approximation to the solution of (2.1) which is valid for $0 \le z \le 1$. The transformation $z = x+1$ transforms (2.1) into an equation of the same form with the initial condition $y_{x=0} = y_{z=1}$. One can now construct an approximation to the solution of the resulting equation, $y_{n,1}$, which is valid for $1 \le z \le 2$. Continuing in this manner, one obtains a sequence of approximations, $y_{n,i}$ each of which is valid for $i \le z \le i+1$.

We remark that if the sequence of Padé approximants (2.5) converge (which is the case if the power series (2.2) converges), one can compute the response to (2.1) to any degree of accuracy, whereas in the other usual approximation procedures, i.e., perturbation schemes, etc., the degree of approximation is of fixed accuracy.

III. INCLUSION OF DAMPING TERM

Here we derive approximations to the response of a dynamical system in the absence of external forces with viscous damping and a nonlinear restoring force.

$$y'' + 2ky' + ay + by^2 + cy^3 = 0 \ , \ y(0) = a_0 \ , \ y'(0) = b_0 \ ,$$

$$0 < 2k \ll 1 \ , \qquad (3.1)$$

and no restrictions are placed on the relative magnitudes of a, b and c.

In the usual perturbation scheme either the nonlinear restoring force or both the damping and nonlinear restoring force are considered to be very small. If the coefficients b and c are not small, this perturbation scheme is not an effective means of obtaining good approximations to the response of (1.1).

We now present an alternative technique which utilizes the approximations developed in Section II.

Let

$$y = e^{-kt} v \quad . \tag{3.2}$$

Then (3.1) becomes

$$v'' + (a-k^2)v + be^{-kt}v^2 + ce^{-2kt}v^3 = 0 \ , \ v(0) = a_o \ , \ v'(0) = ka_o + b_o \ . \tag{3.3}$$

Since k is assumed to be very small, the coefficients in (3.3) are very nearly constant over a small interval of time. We shall assign the constant values B and C to the coefficients of v^2 and v^3 in (3.3), respectively, and construct the approximate solution, w_n , to the resulting equation

$$w'' + (a-k^2)w + Bw^2 + Cw^3 = 0 \ , \ w(0) = a_o \ , \ w'(0) = ka_o + b_o \ , \tag{3.4}$$

over a fixed time interval. The approximate solution to (3.1) is given by

$$y_n = e^{-kt} w_n \quad . \tag{3.5}$$

We can then use the method of analytic continuation as described in Section II to construct the approximations for a wide range of the argument.

We present three examples which illustrate the techniques described in this report. The sixth order Padé approximant is used in all examples.

Let y satisfy

$$y'' + 0.10y' + 10y + 100y^3 = 0 \ , \ y(0) = 1 \ , \ y'(0) = 0 \quad . \tag{3.6}$$

The transformation (3.2) yields

$$v'' + 0.9975v + 100e^{-0.10t}v^3 = 0 \ , \ v(0) = 1 \ , \ v'(0) = 0.05 \quad . \tag{3.7}$$

Now let w be the solution to

$$w'' + 0.9975w + 100w^3 = 0 \ , \ w(0) = 1 \ , \ w'(0) = 0.05 \quad , \tag{3.8}$$

and let w_6 be the sixth order Padé approximation to the solution of (3.8). Table III compares $y(t)$ and $u(t) = e^{-0.05t}w_6(t)$.

TABLE III

t	y(t)	u(t)	y(t)-u(t)
0.00	1.0000	1.0000	0.0000
0.02	0.9782	0.9782	0.0000
0.04	0.9156	0.9155	0.0001
0.06	0.8189	0.8186	0.0003
0.08	0.6972	0.6966	0.0006
0.10	0.5592	0.5582	0.0010
0.12	0.4121	0.4107	0.0014
0.14	0.2606	0.2589	0.0017
0.16	0.1076	0.1054	0.0022
0.18	-0.0456	-0.0484	0.0028
0.20	-0.1983	-0.2022	0.0039
0.22	-0.3495	-0.3557	0.0062
0.24	-0.4971	-0.5081	0.0110
0.26	-0.6374	-0.6580	0.0206
0.28	-0.7643	-0.8030	0.0387
0.30	-0.8700	-0.9402	0.0702

For the second example, let

$$y'' + 0.20y' + 5y + 10y^3 = 0 \ , \ y(0) = 1, \ y'(0) = 0 \quad . \tag{3.9}$$

Then

$$v'' + 4.99v + 10e^{-0.20t}v^3 = 0 \ , \ v(0) = 1 \ , \ v'(0) = 0.10 \quad . \tag{3.10}$$

where $y = e^{-0.10t} v$. We consider the related equation,

$$w'' + 4.99w + Bw^3 = 0 \ , \ w(0) = 1 \ , \ w'(0) = 0.10 \quad . \qquad (3.11)$$

Tables IV and V show that a judicious choice of B in (3.11) results in a very effective approximation. Table IV compares $y(t)$ with $u_1 = e^{-0.10t} w_6(t)$ where $w_6(t)$ is the sixth order Padé approximation to the solution of (3.11) in which $B = 10$. In Table V, $u_2(t) = e^{-0.10t} w_6(t)$ where the value of B is chosen to be the average of $10e^{-0.20t}$ over the interval (0,0.2) .

TABLE IV

t	$y(t)$	$u_1(t)$	$y(t)-u_1(t)$
0.00	1.0000	1.0000	0.0000
0.02	0.9970	0.9970	0.0000
0.04	0.9881	0.9881	0.0000
0.06	0.9734	0.9733	0.0001
0.08	0.9531	0.9530	0.0001
0.10	0.9256	0.9273	-0.0007
0.12	0.8971	0.8966	0.0005
0.14	0.8621	0.8613	0.0008
0.16	0.8230	0.8218	0.0012
0.18	0.7801	0.7786	0.0015
0.20	0.7339	0.7320	0.0019
0.22	0.6850	0.6825	0.0025
0.24	0.6335	0.6305	0.0030
0.26	0.5799	0.5764	0.0035
0.28	0.5247	0.5206	0.0041
0.30	0.4680	0.4634	0.0046
0.32	0.4102	0.4050	0.0052
0.34	0.3516	0.3458	0.0058
0.36	0.2923	0.2859	0.0064
0.38	0.2325	0.2256	0.0069
0.40	0.1725	0.1650	0.0075
0.42	0.1124	0.1044	0.0080
0.44	0.0522	0.0437	0.0085
0.46	-0.0078	-0.0167	0.0089

TABLE V

t	$y(t)$	$u_2(t)$	$y(t)-u_2(t)$
0.00	1.0000	1.0000	0.0000
0.02	0.9970	0.9970	0.0000
0.04	0.9881	0.9882	-0.0001
0.06	0.9734	0.9737	-0.0003
0.08	0.9531	0.9536	-0.0005
0.10	0.9256	0.9282	-0.0026
0.12	0.8971	0.8979	-0.0008
0.14	0.8621	0.8630	-0.0009
0.16	0.8230	0.8240	-0.0010
0.18	0.7801	0.7812	-0.0011
0.20	0.7339	0.7351	-0.0012
0.22	0.6850	0.6861	-0.0011
0.24	0.6335	0.6345	-0.0010
0.26	0.5799	0.5809	-0.0010
0.28	0.5247	0.5255	-0.0008
0.30	0.4680	0.4687	-0.0007
0.32	0.4102	0.4108	-0.0006
0.34	0.3516	0.3519	-0.0003
0.36	0.2923	0.2925	-0.0003
0.38	0.2325	0.2325	0.0000
0.40	0.1725	0.1723	0.0002
0.42	0.1124	0.1120	0.0004
0.44	0.0522	0.0516	0.0006
0.46	-0.0078	-0.0086	0.0008
0.48	-0.0675	-0.0686	0.0009
0.50	-0.1269	-0.1282	0.0013

We close the section by giving an example of the method of analytic continuation. We continue the solution of (3.11) as given in Table IV. Using the tabular value from Table II for $u(t)$ at $t = 0.1$ as the true value, we make the transformation $t = 0.1 + \tau$ in (3.11) and $u(\tau)|_{\tau=0} = u(0.1)$. The table below gives the sixth order Padé approximations to the solution of the resulting equation. Here we choose $B = 10e^{-0.01}$.

TABLE VI

t	y(t)	u(t)	y(t)-u(t)
0.10	0.9276	0.9273	0.0003
0.12	0.8971	0.8968	0.0003
0.14	0.8621	0.8617	0.0004
0.16	0.8230	0.8226	0.0004
0.18	0.7801	0.7797	0.0004
0.20	0.7339	0.7335	0.0004
0.22	0.6849	0.6844	0.0005
0.24	0.6335	0.6328	0.0007
0.26	0.5799	0.5791	0.0008
0.28	0.5247	0.5236	0.0011
0.30	0.4680	0.4668	0.0012
0.32	0.4102	0.4088	0.0014
0.34	0.3516	0.3499	0.0017
0.36	0.2923	0.2904	0.0019
0.38	0.2325	0.2305	0.0020
0.40	0.1725	0.1703	0.0022

Note that these values are much better than those in Table IV. Thus the combination of analytic continuation and judicious selection of the value of B in (3.11) can be expected to yield very good approximations to the solution of (3.10).

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